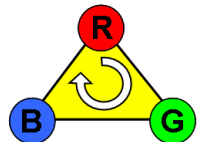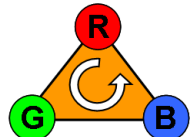# THEOREM OF THE DAY

Sperner's Lemma *Suppose the interior of a triangle is triangulated (that is, divided up internally into small triangles). The vertices of the triangle are coloured red, green and blue, respectively. All other vertices, where lines meet inside or around the outside edges of the triangle, are also coloured red, green or blue with the restriction that no edge of the main triangle contains all three colours. Let T_C be the number of small triangles whose vertices are coloured red, green and blue in* clockwise *order; and let T_A be the number of small triangles whose vertices are coloured red, green and blue in* anticlockwise *order. Then*

$$|T_C - T_A| = 1.$$

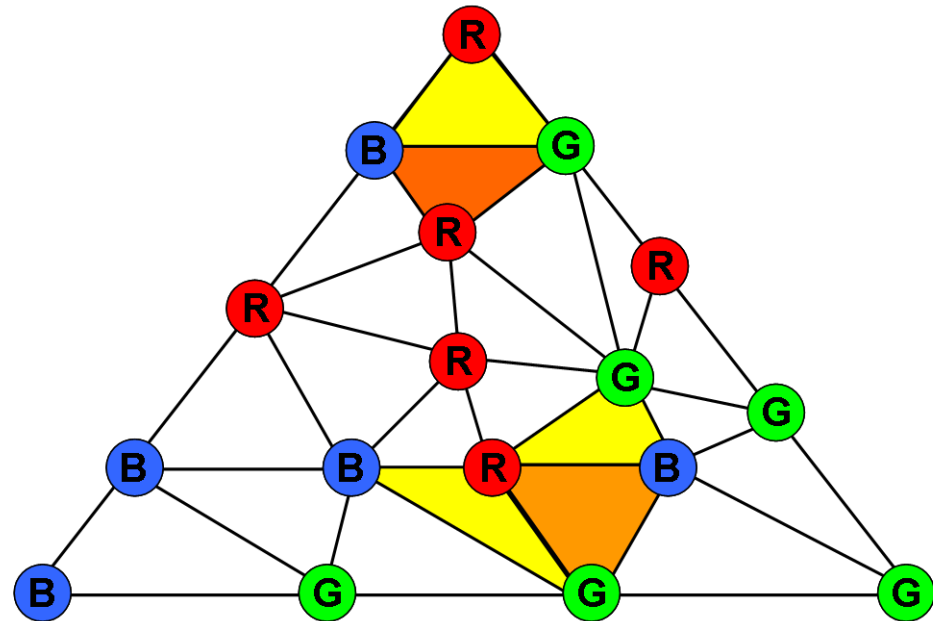*In particular, the total number of small red-green-blue triangles must be odd, and is certainly never zero.*

Counted by T_C Counted by T_A

The triangulation shown on the right has been given a so-called *Sperner colouring*. The small triangles which have vertices of all three colours have been colour-coded as shown above and you can check that $T_C = 3$ and $T_A = 2$. Two people can play a game starting with a large Sperner-coloured triangulation and taking it in turns to spot a small red-green-blue triangle: provided you start you know you are sure to win!

This lemma might appear to be just a mathematical curiosity; in fact, it is equivalent to the powerful and important Brouwer fixed point theorem. Emanuel Sperner (1905–1980) published it in 1928, the year he received his doctorate from the University of Hamburg under the famous group theorist Otto Schreier.

Web link: www.cut-the-knot.org/Curriculum/Geometry/SpernerLemma.shtml

Further reading: *A Combinatorial Introduction to Topology* by Michael Henle, Dover Publications, 1994