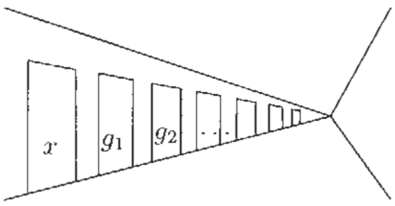
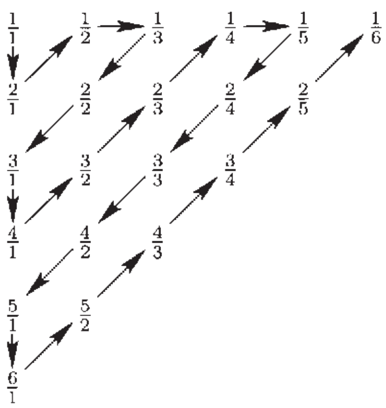


manager's surprise (he is not a mathematician) this works; he can still put up all guests plus the new arrival x!

Now it is clear that he can also put up another guest y, and another one z, and so on. In particular, we note that, in contrast to finite sets, it may well happen that a proper subset of an *infinite* set M has the same size as M. In fact, as we will see, this is a characterization of infinity: A set is infinite if and only if it has the same size as some proper subset.

Let us leave Hilbert's hotel and look at our familiar number sets. The set $\mathbb{Z}$ of integers is again countable, since we may enumerate $\mathbb{Z}$ in the form $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$. It may come more as a surprise that the rationals can be enumerated in a similar way.

Theorem 1. *The set $\mathbb{Q}$ of rational numbers is countable.*



■ **Proof.** By listing the set $\mathbb{Q}^+$ of positive rationals as suggested in the figure in the margin, but leaving out numbers already encountered, we see that $\mathbb{Q}^+$ is countable, and hence so is $\mathbb{Q}$ by listing 0 at the beginning and $-\frac{p}{q}$ right after $\frac{p}{q}$. With this listing

$$\mathbb{Q} = \{0, 1, -1, 2, -2, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{3}, 3, -3, 4, -4, \tfrac{3}{2}, -\tfrac{3}{2}, \ldots\}. \qquad \square$$

Another way to interpret the figure is the following statement:

The union of countably many countable sets M_n is again countable.

Indeed, set $M_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}$ and list

$$\bigcup_{n=1}^{\infty} M_n = \{a_{11}, a_{21}, a_{12}, a_{13}, a_{22}, a_{31}, a_{41}, a_{32}, a_{23}, a_{14}, \ldots\}$$

precisely as before.

Let us contemplate Cantor's enumeration of the positive rationals a bit more. Looking at the figure we obtained the sequence

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{4}, \frac{3}{3}, \frac{4}{2}, \frac{5}{1}, \ldots$$

and then had to strike out the duplicates such as $\frac{2}{2} = \frac{1}{1}$ or $\frac{2}{4} = \frac{1}{2}$.

But there is a listing that is even more elegant and systematic, and which contains no duplicates — found only quite recently by Neil Calkin and Herbert Wilf. Their new list starts as follows:

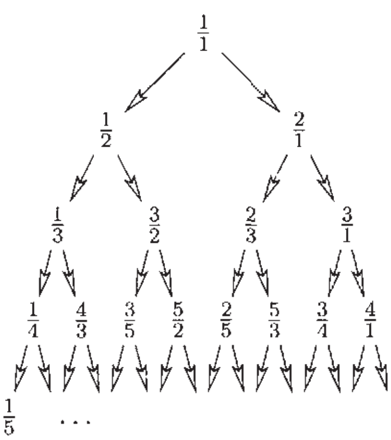
$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \ldots .$$

Here the denominator of the n-th rational number equals the numerator of the $(n+1)$-st number. In other words, the n-th fraction is $b(n)/b(n+1)$, where $\big(b(n)\big)_{n\geq 0}$ is a sequence that starts with

$$(1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, \ldots).$$

This sequence has first been studied by a German mathematician, Moritz Abraham Stern, in a paper from 1858, and is has become known as "Stern's diatomic series."

How do we obtain this sequence, and hence the Calkin-Wilf listing of the positive fractions? Consider the infinite binary tree in the margin. We immediately note its recursive rule:



- $\frac{1}{1}$ is on top of the tree, and
- every node $\frac{i}{j}$ has two sons: the left son is $\frac{i}{i+j}$ and the right son is $\frac{i+j}{j}$.

We can easily check the following four properties:

(1) All fractions in the tree are reduced, that is, if $\frac{r}{s}$ appears in the tree, then r and s are relatively prime.

This holds for the top $\frac{1}{1}$, and then we use induction downward. If r and s are relatively prime, then so are r and $r+s$, as well as s and $r+s$.

(2) Every reduced fraction $\frac{r}{s} > 0$ appears in the tree.

We use induction on the sum $r+s$. The smallest value is $r+s=2$, that is $\frac{r}{s} = \frac{1}{1}$, and this appears at the top. If $r > s$, then $\frac{r-s}{s}$ appears in the tree by induction, and so we get $\frac{r}{s}$ as its right son. Similarly, if $r < s$, then $\frac{r}{s-r}$ appears, which has $\frac{r}{s}$ as its left son.

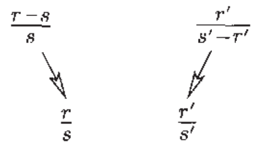
(3) Every reduced fraction appears exactly once.

The argument is similar. If $\frac{r}{s}$ appears more than once, then $r \neq s$, since any node in the tree except the top is of the form $\frac{i}{i+j} < 1$ or $\frac{i+j}{j} > 1$. But if $r > s$ or $r < s$, then we argue by induction as before.

Every positive rational appears therefore exactly once in our tree, and we may write them down listing the numbers level-by-level from left to right. This yields precisely the initial segment shown above.

(4) The denominator of the n-th fraction in our list equals the numerator of the $(n+1)$-st.

This is certainly true for $n = 0$, or when the n-th fraction is a left son. Suppose the n-th number $\frac{r}{s}$ is a right son. If $\frac{r}{s}$ is at the right boundary, then $s = 1$, and the successor lies at the left boundary and has numerator 1. Finally, if $\frac{r}{s}$ is in the interior, and $\frac{r'}{s'}$ is the next fraction in our sequence, then $\frac{r}{s}$ is the right son of $\frac{r-s}{s}$, $\frac{r'}{s'}$ is the left son of $\frac{r'}{s'-r'}$, and by induction the denominator of $\frac{r-s}{s}$ is the numerator of $\frac{r'}{s'-r'}$, so we get $s = r'$.



Well, this is nice, but there is even more to come. There are two natural questions:

– Does the sequence $\big(b(n)\big)_{n\geq 0}$ have a "meaning"? That is, does $b(n)$ count anything simple?

– Given $\frac{r}{s}$, is there an easy way to determine the successor in the listing?

To answer the first question, we work out that the node $b(n)/b(n+1)$ has the two sons $b(2n+1)/b(2n+2)$ and $b(2n+2)/b(2n+3)$. By the set-up of the tree we obtain the recursions

$$b(2n+1) = b(n) \quad \text{and} \quad b(2n+2) = b(n) + b(n+1). \tag{1}$$

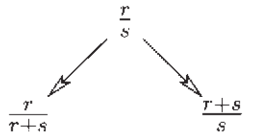
With $b(0) = 1$ the sequence $(b(n))_{n\geq 0}$ is completely determined by (1).

So, is there a "nice" "known" sequence which obeys the same recursion? Yes, there is. We know that any number n can be uniquely written as a sum of distinct powers of 2 — this is the usual binary representation of n. A *hyper-binary* representation of n is a representation of n a sum of powers of 2, where every power 2^k appears at most *twice*. Let $h(n)$ be the number of such representations for n. You are invited to check that the sequence $h(n)$ obeys the recursion (1), and this gives $b(n) = h(n)$ for all n.

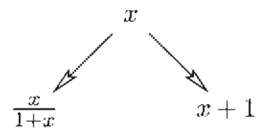
For example, $h(6) = 3$, with the hyper-binary representations
$6 = 4 + 2$
$6 = 4 + 1 + 1$
$6 = 2 + 2 + 1 + 1$.

Incidentally, we have proved a surprising fact: Let $\frac{r}{s}$ be a reduced fraction, there exists precisely one integer n with $r = h(n)$ and $s = h(n+1)$.

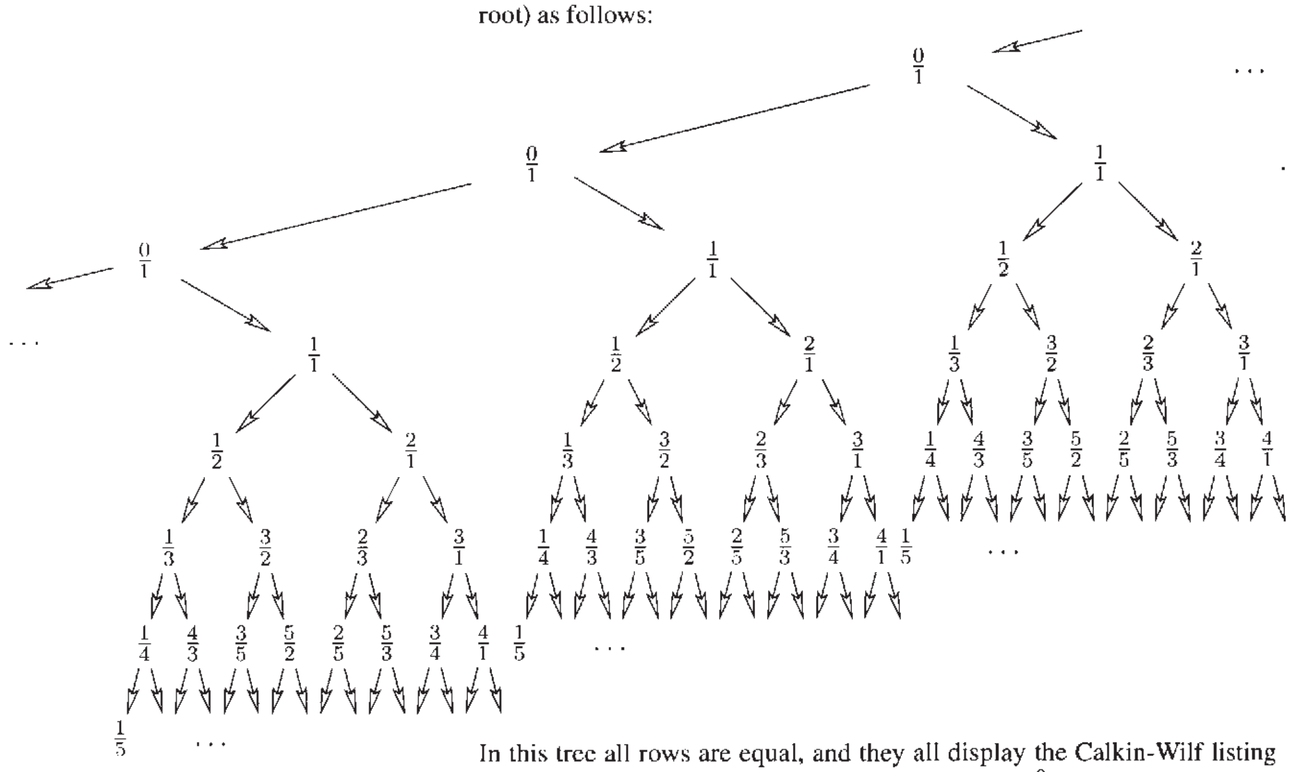
Let us look at the second question. We have in our tree



that is, with $x := \frac{r}{s}$,



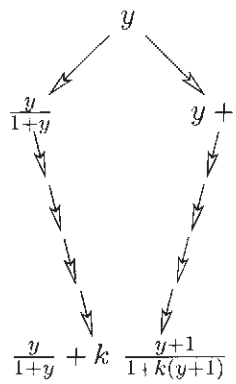
We now use this to generate an even larger infinite binary tree (without a root) as follows:



In this tree all rows are equal, and they all display the Calkin-Wilf listing of the positive rationals (starting with an additional $\frac{0}{1}$).

So how does one get from one rational to the next? To answer this, we first record that for every rational x its right son is $x+1$, the right grand-son is $x+2$, so the k-fold right son is $x+k$. Similarly, the left son of x is $\frac{x}{1+x}$, whose left son is $\frac{x}{1+2x}$, and so on: The k-fold left son of x is $\frac{x}{1+kx}$.
Now to find how to get from $\frac{r}{s} = x$ to the "next" rational $f(x)$ in the listing, we have to analyze the situation depicted in the margin. In fact, if we consider any nonnegative rational number x in our infinite binary tree, then it is the k-fold right son of the left son of some rational $y \geq 0$ (for some $k \geq 0$), while $f(x)$ is given as the k-fold left son of the right son of the same y. Thus with the formulas for k-fold left sons and k-fold right sons, we get

$$x = \frac{y}{1+y} + k.$$



as claimed in the figure in the margin. Here $k = \lfloor x \rfloor$ is the integral part of x, while $\frac{y}{1+y} = \{x\}$ is the fractional part. And from this we obtain

$$f(x) = \frac{y+1}{1+k(y+1)} = \frac{1}{\frac{1}{y+1}+k} = \frac{1}{k+1-\frac{y}{y+1}} = \frac{1}{\lfloor x \rfloor + 1 - \{x\}}.$$

Thus we have obtained a beautiful formula for the successor $f(x)$ of x, found very recently by Moshe Newman:

The function

$$x \longmapsto f(x) = \frac{1}{\lfloor x \rfloor + 1 - \{x\}}$$

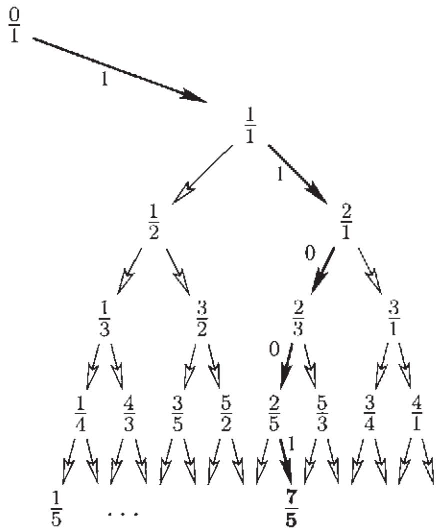
generates the Calkin-Wilf sequence

$$\tfrac{1}{1} \mapsto \tfrac{1}{2} \mapsto \tfrac{2}{1} \mapsto \tfrac{1}{3} \mapsto \tfrac{3}{2} \mapsto \tfrac{2}{3} \mapsto \tfrac{3}{1} \mapsto \tfrac{1}{4} \mapsto \tfrac{4}{3} \mapsto \dots$$

which contains every positive rational number exactly once.

The Calkin-Wilf-Newman way to enumerate the positive rationals has a number of additional remarkable properties. For example, one may ask for a fast way to determine the n-th fraction in the sequence, say for $n = 10^6$. Here it is:

> To find the n-th fraction in the Calkin-Wilf sequence, express n as a binary number $n = (b_k b_{k-1} \dots b_1 b_0)_2$, and then follow the path in the Calkin-Wilf tree that is determined by its digits, starting at $\frac{s}{t} = \frac{0}{1}$. Here $b_i = 1$ means "take the right son," that is, "add the denominator to the numerator," while $b_i = 0$ means "take the left son," that is, "add the numerator to the denominator."



The figure in the margin shows the resulting path for $n = 25 = (11001)_2$: So the 25th number in the Calkin-Wilf sequence is $\frac{7}{5}$. The reader could easily work out a similar scheme that computes for a given fraction $\frac{s}{t}$ (the binary representation of) its position n in the Calkin-Wilf sequence.