Of friends and politicians

Chapter 34

It is not known who first raised the following problem or who gave it its human touch. Here it is:

Suppose in a group of people we have the situation that any pair of persons have precisely one common friend. Then there is always a person (the "politician") who is everybody's friend.

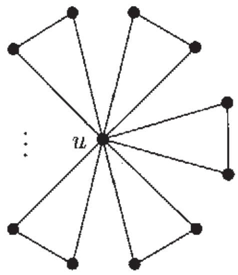
"A politician's smile"

In the mathematical jargon this is called the *friendship theorem*.

Before tackling the proof let us rephrase the problem in graph-theoretic terms. We interpret the people as the set of vertices V and join two vertices by an edge if the corresponding people are friends. We tacitly assume that friendship is always two-ways, that is, if u is a friend of v, then v is also a friend of u, and further that nobody is his or her own friend. Thus the theorem takes on the following form:

Theorem. *Suppose that G is a finite graph in which any two vertices have precisely one common neighbor. Then there is a vertex which is adjacent to all other vertices.*

Note that there are finite graphs with this property; see the figure, where u is the politician. However, these "windmill graphs" also turn out to be the only graphs with the desired property. Indeed, it is not hard to verify that in the presence of a politician only the windmill graphs are possible.



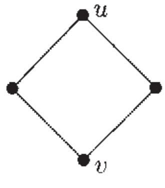
A windmill graph

Surprisingly, the friendship theorem does not hold for infinite graphs! Indeed, for an inductive construction of a counterexample one may start for example with a 5-cycle, and repeatedly add common neighbors for all pairs of vertices in the graph that don't have one, yet. This leads to a (countably) infinite friendship graph without a politician.

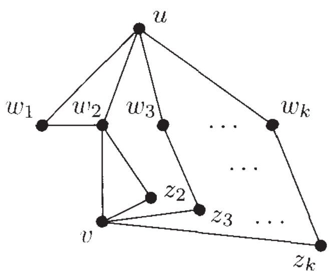
Several proofs of the friendship theorem exist, but the first proof, given by Paul Erdős, Alfred Rényi and Vera Sós, is still the most accomplished.

■ **Proof.** Suppose the assertion is false, and G is a counterexample, that is, no vertex of G is adjacent to all other vertices. To derive a contradiction we proceed in two steps. The first part is combinatorics, and the second part is linear algebra.

(1) We claim that G is a regular graph, that is, $d(u) = d(v)$ for any $u, v \in V$. Note first that the condition of the theorem implies that there are no cycles of length 4 in G. Let us call this the C_4-*condition*.



We first prove that any two *non-adjacent* vertices u and v have equal degree $d(u) = d(v)$. Suppose $d(u) = k$, where $w_1, \ldots, w_k$ are the neighbors of u. Exactly one of the w_i, say w_2, is adjacent to v, and w_2 adjacent to exactly one of the other w_i's, say w_1, so that we have the situation of the figure to the left. The vertex v has with w_1 the common neighbor w_2, and with w_i $(i \geq 2)$ a common neighbor z_i $(i \geq 2)$. By the C_4-condition, all these z_i must be distinct. We conclude $d(v) \geq k = d(u)$, and thus $d(u) = d(v) = k$ by symmetry.



To finish the proof of **(1)**, observe that any vertex different from w_2 is not adjacent to either u or v, and hence has degree k, by what we already proved. But since w_2 also has a non-neighbor, it has degree k as well, and thus G is k-regular.

Summing over the degrees of the k neighbors of u we get k^2. Since every vertex (except u) has exactly one common neighbor with u, we have counted every vertex once, except for u, which was counted k times. So the total number of vertices of G is

$$n = k^2 - k + 1. \tag{1}$$

(2) The rest of the proof is a beautiful application of some standard results of linear algebra. Note first that k must be greater than 2, since for $k \leq 2$ only $G = K_1$ and $G = K_3$ are possible by (1), both of which are trivial windmill graphs. Consider the adjacency matrix $A = (a_{ij})$, as defined on page 220. By part **(1)**, any row has exactly k 1's, and by the condition of the theorem, for any two rows there is exactly one column where they both have a 1. Note further that the main diagonal consists of 0's. Hence we have

$$A^2 = \begin{pmatrix} k & 1 & \ldots & 1 \\ 1 & k & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \ldots & 1 & k \end{pmatrix} = (k-1)I + J,$$

where I is the identity matrix, and J the matrix of all 1's. It is immediately checked that J has the eigenvalues n (of multiplicity 1) and 0 (of multiplicity $n-1$). It follows that A^2 has the eigenvalues $k - 1 + n = k^2$ (of multiplicity 1) and $k - 1$ (of multiplicity $n - 1$).

Since A is symmetric and hence diagonalizable, we conclude that A has the eigenvalues k (of multiplicity 1) and $\pm\sqrt{k-1}$. Suppose r of the eigenvalues are equal to $\sqrt{k-1}$ and s of them are equal to $-\sqrt{k-1}$, with $r + s = n - 1$. Now we are almost home. Since the sum of the eigenvalues of A equals the trace (which is 0), we find

$$k + r\sqrt{k-1} - s\sqrt{k-1} = 0,$$

and, in particular, $r \neq s$, and

$$\sqrt{k-1} = \frac{k}{s-r}.$$

Now if the square root $\sqrt{m}$ of a natural number m is rational, then it is an integer! An elegant proof for this was presented by Dedekind in 1858: Let n_0 be the smallest natural number with $n_0\sqrt{m} \in \mathbb{N}$. If $\sqrt{m} \notin \mathbb{N}$, then there exists $\ell \in \mathbb{N}$ with $0 < \sqrt{m} - \ell < 1$. Setting $n_1 := n_0(\sqrt{m} - \ell)$, we find $n_1 \in \mathbb{N}$ and $n_1\sqrt{m} = n_0(\sqrt{m} - \ell)\sqrt{m} = n_0 m - \ell(n_0\sqrt{m}) \in \mathbb{N}$. With $n_1 < n_0$ this yields a contradiction to the choice of n_0.
Returning to our equation, let us set $h = \sqrt{k-1} \in \mathbb{N}$, then

$$h(s-r) \;=\; k \;=\; h^2 + 1.$$

Since h divides $h^2 + 1$ and h^2, we find that h must be equal to 1, and thus $k = 2$, which we have already excluded. So we have arrived at a contradiction, and the proof is complete. □

However, the story is not quite over. Let us rephrase our theorem in the following way: Suppose G is a graph with the property that between any two vertices there is exactly one path of length 2. Clearly, this is an equivalent formulation of the friendship condition. Our theorem then says that the only such graphs are the windmill graphs. But what if we consider paths of length more than 2? A conjecture of Anton Kotzig asserts that the analogous situation is impossible.

Kotzig's Conjecture. *Let $\ell > 2$. Then there are no finite graphs with the property that between any two vertices there is precisely one path of length ℓ.*

Kotzig himself verified his conjecture for $\ell \leq 8$. In [3] his conjecture is proved up to $\ell = 20$, and A. Kostochka has told us recently that it is now verified for all $\ell \leq 33$. A general proof, however, seems to be out of reach ...

References

[1] P. ERDŐS, A. RÉNYI & V. SÓS: *On a problem of graph theory,* Studia Sci. Math. **1** (1966), 215-235.

[2] A. KOTZIG: *Regularly k-path connected graphs,* Congressus Numerantium **40** (1983), 137-141.

[3] A. KOSTOCHKA: *The nonexistence of certain generalized friendship graphs,* in: "Combinatorics" (Eger, 1987), Colloq. Math. Soc. János Bolyai **52**, North-Holland, Amsterdam 1988, 341-356.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The adjacency matrix for the 5-cycle C_5

Let us carry our discussion a little further. We see from (8) that the larger σ_T is for a representation of G, the better a bound for $\Theta(G)$ we will get. Here is a method that gives us an orthonormal representation for *any* graph G. To $G = (V, E)$ we associate the *adjacency matrix* $A = (a_{ij})$, which is defined as follows: Let $V = \{v_1, \dots, v_m\}$, then we set

$$a_{ij} := \begin{cases} 1 & \text{if } v_iv_j \in E \\ 0 & \text{otherwise.} \end{cases}$$

A is a real symmetric matrix with 0's in the main diagonal.

Now we need two facts from linear algebra. First, as a symmetric matrix, A has m real eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$ (some of which may be equal), and the sum of the eigenvalues equals the sum of the diagonal entries of A, that is, 0. Hence the smallest eigenvalue must be negative (except in the trivial case when G has no edges). Let $p = |\lambda_m| = -\lambda_m$ be the absolute value of the smallest eigenvalue, and consider the matrix

$$M := I + \frac{1}{p}A,$$

where I denotes the $(m \times m)$-identity matrix. This M has the eigenvalues $1+\frac{\lambda_1}{p} \geq 1+\frac{\lambda_2}{p} \geq \dots \geq 1+\frac{\lambda_m}{p} = 0$. Now we quote the second result (the principal axis theorem of linear algebra): If $M = (m_{ij})$ is a real symmetric matrix with all eigenvalues ≥ 0, then there are vectors $\boldsymbol{v}^{(1)}, \dots, \boldsymbol{v}^{(m)} \in \mathbb{R}^s$ for $s = \text{rank}(M)$, such that

$$m_{ij} = \langle \boldsymbol{v}^{(i)}, \boldsymbol{v}^{(j)} \rangle \qquad (1 \leq i, j \leq m).$$

In particular, for $M = I + \frac{1}{p}A$ we obtain

$$\langle \boldsymbol{v}^{(i)}, \boldsymbol{v}^{(i)} \rangle = m_{ii} = 1 \qquad \text{for all } i$$

and

$$\langle \boldsymbol{v}^{(i)}, \boldsymbol{v}^{(j)} \rangle = \frac{1}{p}a_{ij} \qquad \text{for } i \neq j.$$

Since $a_{ij} = 0$ whenever $v_iv_j \notin E$, we see that the vectors $\boldsymbol{v}^{(1)}, \dots, \boldsymbol{v}^{(m)}$ form indeed an orthonormal representation of G.

Let us, finally, apply this construction to the m-cycles C_m for odd $m \geq 5$. Here one easily computes $p = |\lambda_{\min}| = 2\cos\frac{\pi}{m}$ (see the box). Every row of the adjacency matrix contains two 1's, implying that every row of the matrix M sums to $1 + \frac{2}{p}$. For the representation $\{\boldsymbol{v}^{(1)}, \dots, \boldsymbol{v}^{(m)}\}$ this means

$$\langle \boldsymbol{v}^{(i)}, \boldsymbol{v}^{(1)} + \dots + \boldsymbol{v}^{(m)} \rangle = 1 + \frac{2}{p} = 1 + \frac{1}{\cos\frac{\pi}{m}}$$

and hence

$$\langle \boldsymbol{v}^{(i)}, \boldsymbol{u} \rangle = \frac{1}{m}(1 + (\cos\tfrac{\pi}{m})^{-1}) = \sigma$$

for all i. We can therefore apply our main result (8) and conclude

$$\Theta(C_m) \leq \frac{m}{1 + (\cos\frac{\pi}{m})^{-1}} \qquad (\text{for } m \geq 5 \text{ odd}). \tag{9}$$