APPENDIX
HAUSDORFF'S MAXIMALITY THEOREM

FROM:

W. Rudin. Real and Complex Analysis. McGraw-Hill, New York, New York, 3rd edition, 1987.

We shall first prove a lemma which, when combined with the axiom of choice, leads to an almost instantaneous proof of Theorem 4.21.

If $\mathscr{F}$ is a collection of sets and $\Phi \subset \mathscr{F}$, we call Φ a *subchain* of $\mathscr{F}$ provided that Φ is totally ordered by set inclusion. Explicitly, this means that if $A \in \Phi$ and $B \in \Phi$, then either $A \subset B$ or $B \subset A$. The union of all members of Φ will simply be referred to as the *union of* Φ.

Lemma *Suppose $\mathscr{F}$ is a nonempty collection of subsets of a set X such that the union of every subchain of $\mathscr{F}$ belongs to $\mathscr{F}$. Suppose g is a function which associates to each $A \in \mathscr{F}$ a set $g(A) \in \mathscr{F}$ such that $A \subset g(A)$ and $g(A) - A$ consists of at most one element. Then there exists an $A \in \mathscr{F}$ for which $g(A) = A$.*

PROOF Fix $A_0 \in \mathscr{F}$. Call a subcollection $\mathscr{F}'$ of $\mathscr{F}$ a *tower* if $\mathscr{F}'$ has the following three properties:

(*a*) $A_0 \in \mathscr{F}'$.
(*b*) The union of every subchain of $\mathscr{F}'$ belongs to $\mathscr{F}'$.
(*c*) If $A \in \mathscr{F}'$, then also $g(A) \in \mathscr{F}'$.

The family of all towers is nonempty. For if $\mathscr{F}_1$ is the collection of all $A \in \mathscr{F}$ such that $A_0 \subset A$, then $\mathscr{F}_1$ is a tower. Let $\mathscr{F}_0$ be the intersection of all towers. Then $\mathscr{F}_0$ is a tower (the verification is trivial), but no proper subcollection of $\mathscr{F}_0$ is a tower. Also, $A_0 \subset A$ if $A \in \mathscr{F}_0$. The idea of the proof is to show that $\mathscr{F}_0$ is a subchain of $\mathscr{F}$.

Let Γ be the collection of all $C \in \mathscr{F}_0$ such that every $A \in \mathscr{F}_0$ satisfies either $A \subset C$ or $C \subset A$.

For each $C \in \Gamma$, let $\Phi(C)$ be the collection of all $A \in \mathscr{F}_0$ such that either $A \subset C$ or $g(C) \subset A$.

Properties (*a*) and (*b*) are clearly satisfied by Γ and by each $\Phi(C)$. Fix $C \in \Gamma$, and suppose $A \in \Phi(C)$. We want to prove that $g(A) \in \Phi(C)$. If $A \in \Phi(C)$, there are three possibilities: Either $A \subset C$ and $A \neq C$, or $A = C$, or $g(C) \subset A$. If A is a proper subset of C, then C cannot be a proper subset of $g(A)$, otherwise $g(A) - A$ would contain at least two elements; since $C \in \Gamma$, it follows that $g(A) \subset C$. If $A = C$, then $g(A) = g(C)$. If $g(C) \subset A$, then also $g(C) \subset g(A)$, since $A \subset g(A)$. Thus $g(A) \in \Phi(C)$, and we have proved that $\Phi(C)$ is a tower. The minimality of $\mathscr{F}_0$ implies now that $\Phi(C) = \mathscr{F}_0$, for every $C \in \Gamma$.

In other words, if $A \in \mathscr{F}_0$ and $C \in \Gamma$, then either $A \subset C$ or $g(C) \subset A$. But this says that $g(C) \in \Gamma$. Hence Γ is a tower, and the minimality of $\mathscr{F}_0$ shows that $\Gamma = \mathscr{F}_0$. It follows now from the definition of Γ that $\mathscr{F}_0$ is totally ordered.

Let A be the union of $\mathscr{F}_0$. Since $\mathscr{F}_0$ satisfies (*b*), $A \in \mathscr{F}_0$. By (*c*), $g(A) \in \mathscr{F}_0$. Since A is the largest member of $\mathscr{F}_0$ and since $A \subset g(A)$, it follows that $A = g(A)$. ////

Definition A *choice function* for a set X is a function f which associates to each nonempty subset E of X an element of E: $f(E) \in E$.

In more informal terminology, f "chooses" an element out of each nonempty subset of X.

The Axiom of Choice *For every set there is a choice function.*

Hausdorff's Maximality Theorem *Every nonempty partially ordered set P contains a maximal totally ordered subset.*

PROOF Let $\mathscr{F}$ be the collection of all totally ordered subsets of P. Since every subset of P which consists of a single element is totally ordered, $\mathscr{F}$ is not empty. Note that the union of any chain of totally ordered sets is totally ordered.

Let f be a choice function for P. If $A \in \mathscr{F}$, let A^* be the set of all x in the complement of A such that $A \cup \{x\} \in \mathscr{F}$. If $A^* \neq \varnothing$, put

$$g(A) = A \cup \{f(A^*)\}.$$

If $A^* = \varnothing$, put $g(A) = A$.

By the lemma, $A^* = \varnothing$ for at least one $A \in \mathscr{F}$, and any such A is a maximal element of $\mathscr{F}$. ////