HOW MANY KNEADS DOES A SEQUENCE NEED?

BARRY R. SMITH

Abstract. We define "kneading" as a simple operation on finite sequences of positive integers. We answer the titular question and in doing so make a surprise connection with a venerable piece of number theory.

1. Kneading sequences

We *pinch* an end of a finite sequence of positive integers by splitting 1 away from the terminal entry as so

$$(4, 7, 3, 1, \ldots) \mapsto (1, 3, 7, 3, 1, \ldots)$$

or, when the entry is 1 to begin with, by pushing the terminal 1 back onto its neighboring entry

$$(1, 3, 7, 3, 1, \ldots) \mapsto (4, 7, 3, 1, \ldots)$$

We will *knead*[1] a finite sequence of positive integers by

- Popping off the first entry, then
- Pinching both ends of what remains, then
- Placing the popped entry at the end of the result.

For instance, repeated kneading of the sequence $(2, 2, 3, 6)$ gives:

$$\begin{aligned}(2,2,3,6) &\mapsto (1,1,3,5,1,2) \mapsto (4,5,1,1,1,1) \mapsto (1,4,1,1,2,4) \mapsto (1,3,1,1,2,3,1,1)\\ &\mapsto (1,2,1,1,2,3,2,1) \mapsto (1,1,1,1,2,3,3,1) \mapsto (2,1,2,3,4,1) \mapsto (3,3,5,2)\\ &\mapsto (1,2,5,1,1,3) \mapsto (1,1,5,1,1,2,1,1) \mapsto (6,1,1,2,2,1) \mapsto (2,2,3,6)\end{aligned}$$

Suddenly, surprisingly, we are back where we started. Naturally, we ask,

Question 1. *How many kneads does a sequence need to return to itself?*

The seeming simplicity of kneading disguises a complex piece of number theory, and answering this question will require several ingredients and a good amount of stirring. We begin with a partial answer to whet our appetites.

Partial answer. *Every finite sequence of positive integers eventually returns to itself after a finite amount of kneading.*

Let us call a cycle like the one above a *kneading cycle*.

Sequences of length 1 or 2 are degenerate cases, and special kneading rules apply. Kneading fixes the sequences of length 1, and kneading the sequence

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[1] Unrelated to the topological notion of "kneading"

(a, b) will give the sequence $(1, b-2, 1, a)$ when $b \geq 3$ and (b, a) when $b = 1$ or 2.

We can verify the partial answer above with a simple idea: assign an integer to each sequence of positive integers, its *alternant*, and show that

(i) For each positive integer $a \neq 2$, there are only finitely many sequences with alternant a,
(ii) Kneading preserves alternants.

Kneading is an invertible process – simply pop the entry off the back, pinch both ends of what remains, then place the popped entry at the front[2]. Thus, kneading permutes the finitely many sequences with a given alternant, and the partial answer follows when the alternant is not 2. It will be possible also to show that the sequences with alternant 2 are (2), $(1, 2)$, $(2, 1)$, and the sequences $(1, q, 1)$ with q an arbitrary positive integer. A direct check shows each of these returns to itself after 1 or 2 kneads.

To define alternants, we turn to continued fractions. Every rational number $\frac{\alpha}{\beta} > 1$ can be expanded in two ways as a finite simple continued fraction:

$$\frac{\alpha}{\beta} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{\ddots + \cfrac{1}{q_l}}}$$

with positive integer quotients $q_1, \ldots, q_l$. (Switching between the two expansions is accomplished by pinching the right end of this sequence.)

We will denote the numerator of the continued fraction with sequence of quotients $q_1, \ldots, q_l$ by $[q_1, \ldots, q_l]$. Such expressions are called *continuants*.

Definition. The *alternant* of a finite sequence of positive integers $\overrightarrow{q} = (q_1, \ldots, q_l)$ with $l \geq 3$ is the difference

$$[\overrightarrow{q}]^* := [q_1, \ldots, q_l] - [q_2, \ldots, q_{l-1}]$$

We define directly the alternant of (q_1) to be q_1 and of (q_1, q_2) to be $q_1 q_2$.

All sequences that were kneaded at the top have alternant 100. For instance $[2, 2, 3, 6]^* = [2, 2, 3, 6] - [2, 3] = 107 - 7 = 100$. Properties (i) and (ii) imply that for $a \neq 2$, the sequences with alternant a are partitioned into finitely many kneading cycles. There can be more than one cycle, as, for instance, $(2, 50)$ has alternant 100 and generates a cycle disjoint from the earlier one. We naturally ask,

Question 2. *How many disjoint kneading cycles have a given alternant?*

To answer these questions, we enter the realm of binary quadratic forms. A binary quadratic form is, for us, a polynomial $Ax^2 + Bxy + Cy^2$ in indeterminates x and y with integer coefficients. The question of which integers are obtained by inputting integers into a given form has motivated a tremendous amount of mathematics. Famous results include Fermat's Two Squares

[2]Except for the sequences (a), $(1, a)$, and $(2, a)$, which the inverse simply reverses.

Theorem that the prime numbers represented by the form $x^2 + y^2$ are those congruent to 1 modulo 4 and the fact that for each nonsquare number $D > 0$, the Pellian Equation $x^2 - Dy^2 = 4$ has a solution with $y \neq 0$.

A study of general forms begins with the notion of Lagrange and Gauss of *equivalent forms*. Two forms are equivalent if one is transformed into the other by acting upon it with a 2×2 matrix with integer coefficients and determinant 1, that is, a matrix in the group $\mathrm{SL}_2(\mathbb{Z})$. Specifically, a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$ transforms the form $f(x, y)$ into

$$f(\alpha x + \beta y, \gamma x + \delta y),$$

another binary quadratic form. This gives a right action of $\mathrm{SL}_2(\mathbb{Z})$ on the set of binary quadratic forms. The action preserves discriminants, so the forms with fixed discriminant D split into equivalence classes comprising forms that can all be transformed into each other. The first major theorem in the theory of binary quadratic forms is that the number of equivalence classes with given discriminant is finite. For $D > 0$, the number of classes of forms all of which have relatively prime coefficients (*primitive forms*) is a *class number*, a central notion of algebraic number theory.

We define the *length parity* of a finite sequence to be 0 if the number of terms in the sequence is even and 1 if the number is odd. All sequences in a kneading cycle have the same length parity.

Answer to Question 2. *For each pair of integers*[3] (a, s) *with* $a > 0$ *and* $s = 0$ *or* 1*, the number of kneading cycles of positive integers with alternant value* a *and length parity* s *is equal to the number of equivalence classes of binary quadratic forms of positive discriminant* $a^2 \pm (-1)^s \cdot 4$.

The number of cycles with given alternant is essentially[4] a *class number*!

To answer Question 1, we must continue on to *reduction* of forms. A reduction algorithm is a standard method for determining when two forms are equivalent. Reduction is much more complicated in the case of interest to us, when $D > 0$. In fact, it seems to be little known that there are competing notions of reduction in this case. We shall use Zagier reduction [2] rather than the more common reduction of Lagrange and Gauss.

Zagier declares a form $f = Ax^2 + Bxy + Cy^2$ to be *reduced* if

$$A > 0, \qquad C > 0, \qquad B > A + C.$$

To perform a Zagier reduction step on f, we

(i) Compute the "reducing number", determined as the unique integer n satisfying

$$n - 1 < \frac{B + \sqrt{D}}{2A} < n,$$

[3] excluding $(a, s) = (1, 1)$ and $(2, 1)$

[4] Technically, the number of kneading cycles matches the number of classes of forms, imprimitive and primitive. But when the discriminant is square free, the number of kneading cycles is a bona fide class number.

in which D is the discriminant of f.

(ii) Act on the form f with the matrix

$$\begin{bmatrix} n & 1 \\ -1 & 0 \end{bmatrix}$$

Zagier reduction is iteration of reduction steps.

Because for a reduced form $D \geq D - (B-2A)^2 = 4A(B-A-C) > 0$, we see that the reduced forms with given discriminant D have bounded A. The same inequalities then imply that B must be bounded, hence C must be as well. There are thus finitely many Zagier-reduced forms with given positive discriminant. Zagier shows that every form will reach a reduced form after finitely many reduction steps, after which it will continue through a cycle of reduced forms. He also shows that two reduced forms are equivalent if and only if each can be obtained from the other by reduction, that is, both will be in the same cycle of forms. Thus, every equivalence class contains reduced forms, and the number of cycles of reduced primitive forms is a class number.

For integers $a > 0$ and $s = 0$ or 1, excepting the cases $(a, s) = (1, 1)$ and $(2, 1)$, we define sets

$$S_{(a,s)} = \{\, \text{Sequences of positive integers with alternant } a \text{ and length parity } s \,\}$$

$$Z_{(a,s)} = \{\, \text{Zagier-reduced forms of discriminant } a^2 + (-1)^s \cdot 4 \,\}$$

We define a map $\psi_{(a,s)} \colon Z_{(a,s)} \to S_{(a,s)}$ as follows. If $f = Ax^2 + Bxy + Cy^2$ is in $Z_{(a,s)}$, we first compute $z = (a+B)/2$ (an integer) and expand the rational number $\frac{z}{A}$ into the unique continued fraction with sequence of quotients of length parity s. We set $\psi_{(a,s)}(f)$ to be this sequence of quotients. It will be shown to have alternant a in Section 2

We also define a map $\phi_{(a,s)} \colon S_{(a,s)} \to Z_{(a,s)}$. Recall the notation $[q_1, \ldots, q_l]$ for a continuant, that is, the numerator of the continued fraction with sequence of quotients $q_1, \ldots, q_l$. We define $\phi_{(a,s)}((q_1, \ldots, q_l))$ to be the form

$$[q_2, \ldots, q_l]\, x^2 + ([q_1, \ldots, q_l] + [q_2, \ldots, q_{l-1}])\, xy + [q_1, \ldots, q_{l-1}]\, y^2 \tag{1}$$

(When $l = 1$, we should interpret this as $\phi_{(a,s)}((q_1)) = x^2 + q_1 xy + y^2$.) In Section 2, the discriminant of the form (1) will be computed as $a^2 + (-1)^s \cdot 4$, where a is the alternant of $(q_1, \ldots, q_l)$.

We will also show:

Theorem 1. *The maps $\psi_{(a,s)}$ and $\phi_{(a,s)}$ are inverses, and through them Zagier reduction of forms corresponds to kneading.*

Properties (i)-(ii) of alternants and the Answer to Question 2 are immediate consequences. We can also now answer the titular question:

Answer to Question 1. *The number of kneads needed for a sequence to return to itself is the length of the cycle containing the corresponding reduced form.*

Example. Consider the form $f = 44x^2 + 114xy + 17y^2$, which has discriminant $100^2 + 4$. To compute the corresponding sequence (with $a = 100$ and $s = 0$), we compute $z = (114+100)/2 = 107$, then expand $\frac{107}{44}$ as a continued fraction with even length

$$\frac{107}{44} = 2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{6}}}$$

Thus, $\psi_{(100,0)}(f) = (2,2,3,6)$, the sequence from the example at the beginning. We reduce f by computing the integer n for which $n-1 < \frac{114+\sqrt{10004}}{88} < n$, that is, $n = 3$, and then act on f by the matrix $\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$ to obtain the new form $f' = 71x^2 + 150xy + 44y^2$. To find $\psi_{(100,0)}(f')$, we calculate $z = (150+100)/2 = 125$, then expand $\frac{125}{71}$ as a continued fraction to obtain the sequence $\psi(f') = (1,1,3,5,1,2)$, the result of kneading $(2,2,3,6)$.

Theorem (1) provides a very efficient method for producing all sequences with given alternant and length parity from a known list of Zagier-reduced forms of a certain discriminant. Alternatively, from a known list of sequences with given alternant, we can compute the entire list of corresponding Zagier-reduced forms. For instance, it can be shown that all sequences with even length and alternant 11 lie in one kneading cycle, namely

$$\begin{aligned}&(1,11) \mapsto (1,9,1,1) \mapsto (1,8,2,1) \mapsto (1,7,3,1) \mapsto (1,6,4,1) \mapsto (1,5,5,1)\\ &\qquad \mapsto (1,4,6,1) \mapsto (1,3,7,1) \mapsto (1,2,8,1) \mapsto (1,1,9,1) \mapsto (11,1) \mapsto (1,11)\end{aligned}$$

From these and (1), we obtain the entire list of Zagier reduced forms of discriminant 125, listed as a reduction cycle (representing the form $Ax^2 + Bxy + Cy^2$ by (A,B,C))

$$\begin{aligned}&(11,13,1) \mapsto (19,31,11) \mapsto (25,45,19) \mapsto (29,55,25) \mapsto (31,61,29) \mapsto (31,63,31)\\ &\mapsto (29,61,31) \mapsto (25,55,29) \mapsto (19,45,25) \mapsto (11,31,19) \mapsto (1,13,11) \mapsto (11,13,1)\end{aligned}$$

Lemmermeyer [1] notes that often the middle terms of the triples in a cycle of Zagier-reduced forms steadily increase until they reach a maximum and then steadily decrease until they return to the minimum. In the above example, this phenomenon is illuminated by the clear pattern in the corresponding kneading cycle. Interestingly, this increasing/decreasing pattern sometimes holds for cycles with discriminants that do not have the form $a^2 \pm 4$.

2. Proof Pudding

There is much to prove. To begin we develop some properties of continued fractions and continuants.

Beginning with $[\cdot] = 1$, $[q_1] = q_1$, continuants satisfy the recurrences

$$(2) \qquad \begin{aligned}[][q_1,\dots,q_l] &= q_1\,[q_2,\dots,q_l] + [q_3,\dots,q_l] \quad \text{or}\\ [q_1,\dots,q_l] &= q_l\,[q_1,\dots q_{l-1}] + [q_1,\dots,q_{l-2}]\end{aligned}$$

We adopt, for now, the first as our definition and later show that it gives the numerator of an appropriate continued fraction. The equivalence with the second recurrence and all other properties we will need follow elegantly from the matrix identity:

$$\begin{bmatrix} q_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_2 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} q_l & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} [q_1, \ldots, q_l] & [q_1, \ldots, q_{l-1}] \\ [q_2, \ldots, q_l] & [q_2, \ldots q_{l-1}] \end{bmatrix}, \tag{3}$$

which can be verified by induction using the first recursion (2).

Transposing both sides of (3) reveals the surprising symmetry $[q_1, \ldots, q_l] = [q_l, \ldots, q_1]$, from which follows the second recursion (2). Taking determinants in (3) yields another useful identity

$$[q_1, \ldots, q_l]\,[q_2, \ldots, q_{l-1}] - [q_1, \ldots, q_{l-1}]\,[q_2, \ldots, q_l] = (-1)^l \tag{4}$$

We also note the simplifications

$$[q_1, q_2, \ldots, q_i, 0, q_{i+1}, \ldots, q_l] = [q_1, q_2, \ldots, q_{i-1}, q_i + q_{i+1}, q_{i+2}, q_l] \tag{5}$$

$$[0, q_1, \ldots, q_l] = [q_2, \ldots, q_l] \tag{6}$$

$$[1, q_1, \ldots, q_l] = [q_1 + 1, \ldots, q_l,] \tag{7}$$

The first follows from (3) and the computation

$$\begin{bmatrix} q_i & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_{i+1} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} q_i + q_{i+1} & 1 \\ 1 & 0 \end{bmatrix};$$

and the others follow readily from the recursion (2).

Now let us return to continued fractions. We can prove inductively that

$$q_1 + \cfrac{1}{q_2 + \cfrac{1}{\ddots + \cfrac{1}{q_l}}} = \frac{[q_1, \ldots, q_l]}{[q_2, \ldots, q_l]} \tag{8}$$

We see from (4) that this fraction is in lowest terms, so the continuant $[q_1, \ldots, q_l]$ is the numerator when the continued fraction with partial quotients $q_1, \ldots, q_l$ is fully simplified.

Now we prove that for given integers $a > 0$ and $s = 0$ or 1 with $(a, s) \neq (1, 1)$ or $(2, 1)$, in order:

(i) If $(q_1, \ldots, q_l)$ has alternant a, then $\phi_{(a,s)}((q_1, \ldots, q_l))$ is Zagier-reduced with discriminant $a^2 + (-1)^l \cdot 4$

(ii) $\psi_{(a,s)} \circ \phi_{(a,s)}$ is the identity map on $S_{(a,s)}$,

(iii) If f is a form of discrimimant $a^2 + (-1)^s \cdot 4$, then $\psi_{(a,s)}(f)$ has alternant a and length parity s,

(iv) $\phi_{(a,s)} \circ \psi_{(a,s)}$ is the identity map on $Z_{(a,s)}$,

(v) Kneading corresponds to Zagier reduction of forms

(i): This is easily verified when $l = 1$ or 2, so let $l \geq 3$. Let $(q_1, \ldots, q_l)$ be a sequence of positive integers with alternant a, and let $\phi_{(a,s)}((q_1, \ldots, q_l)) = Ax^2 + Bxy + Cy^2$ be the form (1). To see that it is reduced, note that coefficients A and C are clearly positive, so we need only check that $B > A + C$. Using (2), we compute

$$\begin{aligned} B - C &= (q_l - 1)\,[q_1, \ldots, q_{l-1}] + [q_1, \ldots, q_{l-2}] + [q_2, \ldots, q_{l-1}] \\ &> (q_l - 1)\,[q_2, \ldots, q_{l-1}] + [q_2, \ldots, q_{l-2}] + [q_2, \ldots, q_{l-1}] \\ &= q_l\,[q_2, \ldots, q_{l-1}] + [q_2, \ldots, q_{l-2}] = [q_2, \ldots, q_l] = A \end{aligned}$$

For the discriminant, we compute, using (4) and (2),

$$\begin{aligned} &([q_1, \ldots, q_l] + [q_2, \ldots, z_{l-1}])^2 - 4\,[q_2, \ldots, z_l]\,[q_1, \ldots, q_{l-1}] \\ &= [q_1, \ldots, q_l]^2 + [q_2, \ldots, q_{l-1}]^2 - 2\,[q_2, \ldots, q_l]\,[q_1, \ldots, q_{l-1}] + (-1)^l \cdot 2 \\ &= (q_1\,[q_2, \ldots, q_l])^2 + 2q_1\,[q_2, \ldots, q_l]\,[q_3, \ldots, q_l] + [q_3, \ldots, q_l]^2 + [q_2, \ldots, q_{l-1}]^2 \\ &\qquad - 2q_1\,[q_2, \ldots, q_l]\,[q_2, \ldots, q_{l-1}] - 2\,[q_2, \ldots, q_l]\,[q_3, \ldots, q_{l-1}] + (-1)^l \cdot 2 \\ &= (q_1\,[q_2, \ldots, q_l] - [q_2, \ldots, q_{l-1}] + [q_3, \ldots, q_l])^2 + (-1)^l \cdot 2 \\ &\qquad - (2\,[q_2, \ldots, q_l]\,[q_3, \ldots, q_{l-1}] - 2\,[q_2, \ldots, q_{l-1}]\,[q_3, \ldots, q_l]) \\ &= a^2 + (-1)^l \cdot 4 \end{aligned}$$

(ii): The definition of alternants and (1) show that the sequence $\psi_{(a,s)}\left(\phi_{(a,s)}((q_1, \ldots, q_l))\right)$ is obtained by expanding in a continued fraction the rational number with denominator $[q_2, \ldots, q_l]$ and numerator $[q_1, \ldots, q_l]$. From (8) and the well-known uniqueness of continued fraction expansions, this sequence is $(q_1, \ldots, q_l)$ (which has the right length parity).

(iii): Choose a Zagier-reduced form $f = Ax^2 + Bxy + Cy^2$ of discriminant $D = a^2 + (-1)^s \cdot 4$ with $a > 2$, $s = 0$ or 1, and $D > 0$. By design, the length parity of $\psi_{(a,s)}(f)$ is s, so we need only worry about the alternant.

First, B and D have the same parity, hence a and B do. The positive integer $z = (a + B)/2$ is thus a divisor of

$$\frac{B^2 - a^2}{4} = AC + \frac{D}{4} - \frac{a^2}{4} = AC + (-1)^s$$

Thus, A is relatively prime to z and $AC \equiv (-1)^{s+1} \pmod{z}$.

Note as well that $a^2 + (-1)^s \cdot 4 = B^2 - 4AC > (A - C)^2$ since f is reduced. Then $a > |A - C|$ since $a > 2$, so $a + A > C$. Hence, using again that f is reduced, we have $z > (a + A + C)/2 > C$.

Expand z/A as a simple continued fraction with sequence of quotients $(q_1, \ldots q_l)$, and choose the length so that l and s have the same parity. From (8), $z = [q_1, \ldots, q_l]$ and $A = [q_2, \ldots, q_l]$. Since $AC \equiv (-1)^{s+1} \pmod{z}$, we also have from (4) the congruence $C \equiv [q_1, \ldots, q_{l-1}] \pmod{z}$. Since $0 < C < z$, it follows that $C = [q_1, \ldots, q_{l-1}]$.

From (4), we have

$$[q_2, \ldots, q_{l-1}] = 2\frac{AC + (-1)^l}{a+B} = \frac{B^2 - (a^2 + (-1)^l \cdot 4) + (-1)^l \cdot 4}{2(a+B)}$$
$$= \frac{B-a}{2}$$

Thus, the alternant of $\psi_{(a,s)}(f)$ is

$$[q_1, \ldots, q_l] - [q_2, \ldots, q_{l-1}] = \frac{B+a}{2} - \frac{B-a}{2} = a$$

(iv): Let $f = Ax^2 + Bxy + Cy^2$ be as in (iii). The verification of (iii) shows at least that the form $\phi_{(a,s)} \circ \psi_{(a,s)}(f)$ is $Ax^2 + B'xy + Cy^2$ for some integer B'. Also, (i) and (iii) show that $B'^2 - 4AC = a^2 + (-1)^s \cdot 4$. But B is the unique such integer, thus $\phi_{(a,s)} \circ \psi_{(a,s)}(f) = f$.
(v): Suppose that $(q_1, \ldots, q_l)$ is a sequence with alternant a and length parity s. The reducing number for Zagier reduction of $\phi_{(a,s)}((q_1, \ldots, q_l))$ is

$$\left\lceil \frac{[q_1, \ldots, q_l] + [q_2, \ldots, q_{l-1}] + \sqrt{D}}{2\,[q_2, \ldots, q_l]} \right\rceil,$$

where $D = a^2 + (-1)^s \cdot 4$ is the discriminant. When $l = 1$, so $q_1 > 2$, the number inside the ceiling is $\frac{q_1 + \sqrt{q_1^2 - 4}}{2}$, making the value of the ceiling q_1. When $l = 2$, the reducing number is

$$\left\lceil \frac{q_1 q_2 + 2 + \sqrt{(q_1 q_2)^2 + 4}}{2q_2} \right\rceil$$

A little algebra shows that this ceiling is $q_1 + 1$ when $q_2 \geq 2$ and $q_1 + 2$ when $q_2 = 1$. A direct check shows that in these cases, reducing the form using the appropriate matrix corresponds to kneading the corresponding sequence. It also explains the special cases required in the definition of kneading.

Otherwise, for $l \geq 3$ the term $\sqrt{D}$ in the numerator is approximately a, so the whole numerator is approximately

$$2\,[q_1, \ldots, q_l] = 2q_1\,[q_2, \ldots, q_l] + 2\,[q_3, \ldots, q_l]\,.$$

More algebra shows that the exact quotient is between q_1 and $q_1 + 1$, so the reducing number is always $q_1 + 1$ in this case. Acting on $\phi_{(a,s)}((q_1, \ldots, q_l))$ by the reduction matrix $\begin{pmatrix} q_1 + 1 & 1 \\ -1 & 0 \end{pmatrix}$, the theorem follows by checking the formulas ((5) shows these are appropriate even when q_2 or q_l is 1)

(9)

$$[q_2-1,q_3,\dots,q_{l-1},q_l-1,1,q_1]$$
$$=(q_1+1)^2\,[q_2,\dots,q_l]-(q_1+1)\left([q_1,\dots,q_l]+[q_2,\dots,q_{l-1}]\right)+[q_1,\dots,q_{l-1}]\,,$$

(10)

$$[1,q_2-1,q_3,\dots,q_{l-1},q_l-1,1,q_1]+[q_2-1,q_3,\dots,q_{l-1},q_l-1,1]$$
$$=(2q_1+2)\,[q_2,\dots,q_l]-\left([q_1,\dots,q_l]+[q_2,\dots,q_{l-1}]\right)$$

(11)

$$[1,q_2-1,q_3,\dots,q_{l-1},q_l-1,1]=[q_2,\dots,q_l]$$

First separating off a q_1 from the second and fourth continuants on the right side of (9), then repeatedly applying (2) simplifies it to

$$\begin{aligned}
&(q_1+1)\left([q_2,\dots,q_l]-[q_3,\dots,q_l]\right)-[q_2,\dots,q_{l-1}]+[q_3,\dots,q_{l-1}]\\
&=(q_2-1)(q_1+1)\,[q_3,\dots,q_l]+(q_1+1)\,[q_4,\dots,q_l]\\
&\qquad-(q_2-1)\,[q_3,\dots,q_{l-1}]-[q_4,\dots,q_{l-1}]\\
&=(q_1+1)\,[q_2-1,q_3,\dots,q_l]-[q_2-1,q_3,\dots,q_{l-1}]\\
&=(q_1+1)\,[q_2-1,q_3,\dots,q_{l-1},q_l-1]+q_1\,[q_2-1,q_3,\dots,q_{l-1}]\\
&=q_1\,[q_2-1,q_3,\dots,q_{l-1},q_l-1,1]+[q_2-1,q_3,\dots,q_{l-1},q_l-1]\\
&=[q_2-1,q_3,\dots,q_{l-1},q_l-1,1,q_1]
\end{aligned}$$

With this, (9) is verified. The verification of (10) is similar, but shorter, after first simplifying the left side to

$$q_1\,[q_2,\dots,q_l]+[q_2,\dots,q_{l-1},q_l-1]+[q_2-1,q_3,\dots,q_l]$$

Equation (11) follows immediately from (7).

Let us conclude with two more questions:

(1) Alternants and the form (1) are defined when the sequence $(q_1,\dots,q_l)$ is in a commutative ring with identity. In particular, using (1) we obtain binary quadratic forms from arbitrary integer sequences, but they are typically not reduced. Can kneading be extended to arbitrary integer sequences in such a way that it still corresponds to Zagier reduction?

(2) The theory binary quadratic forms is greatly enriched by Gauss composition, a composition law that makes the classes a group. We can transfer this group operation to kneading cycles, but is there a more natural definition?

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E-mail address: `barsmith@lvc.edu`

Department of Mathematical Sciences, Lebanon Valley College, Annville, PA, USA