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Note

Every Planar Graph Is 5-Choosable

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We prove the statement of the title, which was conjectured in 1975 by V. G. Vizing and, independently, in 1979 by P. Erdös, A. L. Rubin, and H. Taylor.

A list coloring of a graph G is an assignment of colors to the vertices such that adjacent vertices get distinct colors and such that each vertex v receives a color in a prescribed list $L(v)$ of colors. G is *k-choosable* if such a coloring always exists provided that each $L(v)$ has k colors.

In 1975 Vizing raised the question whether every planar graph is 5-choosable (see [2]). Erdös *et al.* [1] conjectured that every planar graph is 5-choosable, but not necessarily 4-choosable. Recently, Voigt [3] described planar graphs that are not 4-choosable. In this paper we prove that they are always 5-choosable. The trick is to find an appropriate extension. The proof is probably the simplest proof of the 5-color theorem for planar graphs.

THEOREM. *Let G be a near-triangulation; i.e., G is a planar graph which has no loops or multiple edges and which consists of a cycle $C: v_1v_2 \cdots v_pv_1$, and vertices and edges inside C such that each bounded face is bounded by a triangle. Assume that v_1 and v_2 are colored* 1 *and* 2, *respectively, and that $L(v)$ is a list of at least three colors if $v \in C - \{v_1, v_2\}$ and at least five colors if $v \in G - C$. Then the coloring of v_1 and v_2 can be extended to a list coloring of G.*

Proof (by induction on the number of vertices of G). If $p = 3$ and $G = C$ there is nothing to prove. So we proceed to the induction step.

If C has a chord v_iv_j, where $2 \leqslant i \leqslant j-2 \leqslant p-1$ ($v_{p+1} = v_1$), then we apply the induction hypothesis to the cycle $v_1v_2 \cdots v_iv_jv_{j+1} \cdots v_1$ and its interior and then to $v_jv_iv_{i+1} \cdots v_{j-1}v_j$ and its interior. So we can assume that C has no chord.

Let $v_1, u_1, u_2, ..., u_m, v_{p-1}$ be the neighbors of v_p in that clockwise order around v_p. As the interior of C is triangulated, G contains the path $P: v_1u_1u_2 \cdots u_mv_{p-1}$. As C is chordless, $P \cup (C - v_p)$ is a cycle C'. Let x, y be two distinct colors in $L(v_p) \backslash \{1\}$. Now define $L'(u_i) = L(u_i) \backslash \{x, y\}$ for $1 \leqslant i \leqslant m$ and $L'(v) = L(v)$ if v is a vertex of G not in $\{u_1, u_2, ..., u_m\}$. Then we apply the induction hypothesis to C' and its interior and the new list L'. We complete the coloring by assigning x or y to v_p such that v_p and v_{p-1} get distinct colors. ∎

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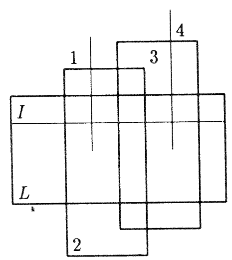

Fig. 3.

be in the interior of P, contradicting the maximality of T_3. This ends the proof of Theorem 1.

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Communication

List colourings of planar graphs

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Abstract

A graph $G=G(V,E)$ is called L-*list colourable* if there is a vertex colouring of G in which the colour assigned to a vertex v is chosen from a list $L(v)$ associated with this vertex. We say G is *k-choosable* if all lists $L(v)$ have the cardinality k and G is L-list colourable for all possible assignments of such lists. There are two classical conjectures from Erdős, Rubin and Taylor 1979 about the choosability of planar graphs:

(1) every planar graph is 5-choosable and,
(2) there are planar graphs which are not 4-choosable.

We will prove the second conjecture.

1. Introduction

There are some generalizations and variations of ordinary graph colourings which are motivated by practical applications ([6]).

For example, it is often required to choose a colour for a vertex v from a *list* $L(v)$ of allowed colours. A graph $G=G(V,E)$ is called *L-list colourable* if there is a colouring f of vertices of G with:

(1) $f(u)\neq f(v) \quad \forall (u,v)\in E(G)$,
(2) $f(v)\in L(v) \quad \forall v\in V(G)$.

G is called *k-choosable* if G is L-list-colourable for every assignment of lists $L(v)$ where each $L(v)$ has exactly k elements.

The idea of L-list colouring, choosability and choice number (the smallest k so that G is k-choosable) was introduced by Erdős, Rubin and Taylor 1979 ([3]). This topic

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has also been studied by Lovász [4], Albertson and Berman [1], Tesman [7], Mahadev, Roberts and Santhanakrishnan [5], Alon and Tarsi [2]. There have also been numerous investigations about similar ideas for edge colourings.

Planar graphs and especially the four-colour problem play an important part in graph theory. Now we are concerned with the choosability of graphs generalizing the ordinary colouring and again the class of planar graphs is very interesting. It is easy to see that every planar graph is 6-choosable and Alon and Tarsi [2] showed that every planar bipartite graph is 3-choosable. This limit is sharp because there are planar bipartite graphs which are not 2-choosable ([3]). Furthermore, there are two intriguing conjectures from Erdős, Rubin and Taylor 1979 [3]:

(1) Every planar graph is 5-choosable.

(2) There are planar graphs which are not 4-choosable.

In the following, we will prove the second conjecture by constructing a planar graph which is not 4-choosable.

2. Construction and list assignment

Remark. (1) In the following, the colours are denoted by numbers: $1, 2, 3, \ldots$.

(2) Most of the specifications in Fig. 1 and Fig. 2 are important only for the later proof.

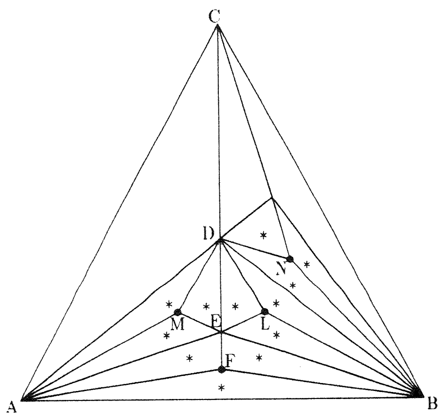

Fig. 1. G_1

The basic graph of the construction is the graph G_1 in Fig. 1. We assign the list $L(v) = \{1, 2, 3, 4\}$ to each vertex of G_1. 12 of the triangles are marked by $*$ and 4 vertices are marked by $\bullet$.

We consider Figs. 2, 3 and 4 which differ only in their list assignments. We insert Fig. 3 into triangle APF and Fig. 4 into triangle BFP: thus, we obtain a triangular figure Δ with 19 inner vertices. Vertex F of Δ is marked $\bullet$.

Next we insert Δ into each of the 12 marked triangles in Fig. 1 such that each vertex marked $\bullet$ in Fig. 1 is identified with the marked vertex in the respective copy of Δ.

Consequently, the resulting graph G_p has $10 + 12 * 19 = 238$ vertices.

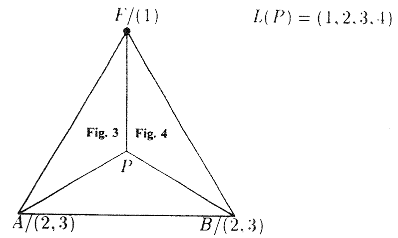

Fig. 2.

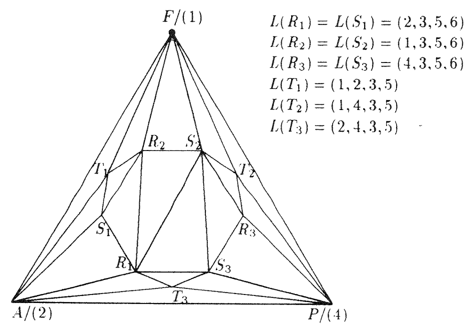

Fig. 3.

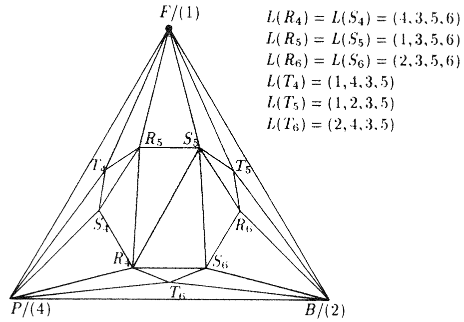

Fig. 4.

3. The theorem and its proof

Theorem. *The planar graph G_p constructed in the previous section is not 4-choosable.*

Proof. We assume there is a L-list colouring f of G_p for the given list assignment.

Lemma. *One of the marked vertices F, M, L, N in Fig. 1 is coloured with colour 1.*

Proof of the Lemma. We consider the K_4: $ABCD$. The assigned lists are $L(A) = L(B) = L(C) = L(D) = (1, 2, 3, 4)$. Consequently, one of these vertices is coloured with colour 1.

(1) $f(A) = 1$.

The vertices A, B, D, E form a K_4. Thus, B, D, and E are coloured with 2, 3 and 4. Consequently, the vertex L has the colour 1.

(2) $f(B) = 1$.

We obtain in an analogous way $f(M) = 1$.

(3) $f(C) = 1$.

We obtain in an analogous way $f(N) = 1$.

(4) $f(D) = 1$.

We obtain in an analogous way $f(F) = 1$. □

Without loss of generality we assume that $f(F) = 1$. Obviously, one of the triangles AFE, EFB, AFB is coloured with the colours 1, 2 and 3. In the following, we suppose (w.l.o.g.):

$$f(F) = 1, \qquad f(A) \in \{2, 3\}, \qquad f(B) \in \{2, 3\}.$$

Considering Fig. 2 we obtain $f(P) = 4$.

(1) $f(A) = 2$.

We use Fig. 3: Because of $f(F) = 1$, $f(A) = 2$ and $f(P) = 4$ it follows immediately:

$$\{f(R_1), f(S_1), f(R_2), f(S_2), f(R_3), f(S_3)\} = \{3, 5, 6\}.$$

Observation. One of the edges (this means their two vertices) S_1R_2, S_2R_3, S_3R_1 is coloured with colours 3 and 5.

The proof is trivial: consider the hexagon $R_1S_1R_2S_2R_3S_3$.

Consequently, one of the vertices T_1, T_2, T_3 is not colourable in accordance with the given lists.

(2) $f(B) = 2$.

Using Fig. 4 we obtain in an analogous way: one of the vertices T_4, T_5, T_6 is not colourable.

This completes the proof of the theorem. □

In addition, it is possible to use other basic graphs instead of G_1 in Fig. 1. It suffices that there is a set of triangles in the graph so that one of these triangles has a fixed colouring, C say. Then we can insert Δ into each of these triangles and assign colour lists to the inner vertices of Δ conflicting with C. In this way, we can construct arbitrarily many planar graphs which are not 4-choosable. Naturally, it is very interesting to find a planar graph which is not 4-choosable and has the minimum number of vertices. Perhaps the graph G_p is already such a graph.

However, the other conjecture dating from 1979 that every planar graph is 5-choosable remains an open problem.

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