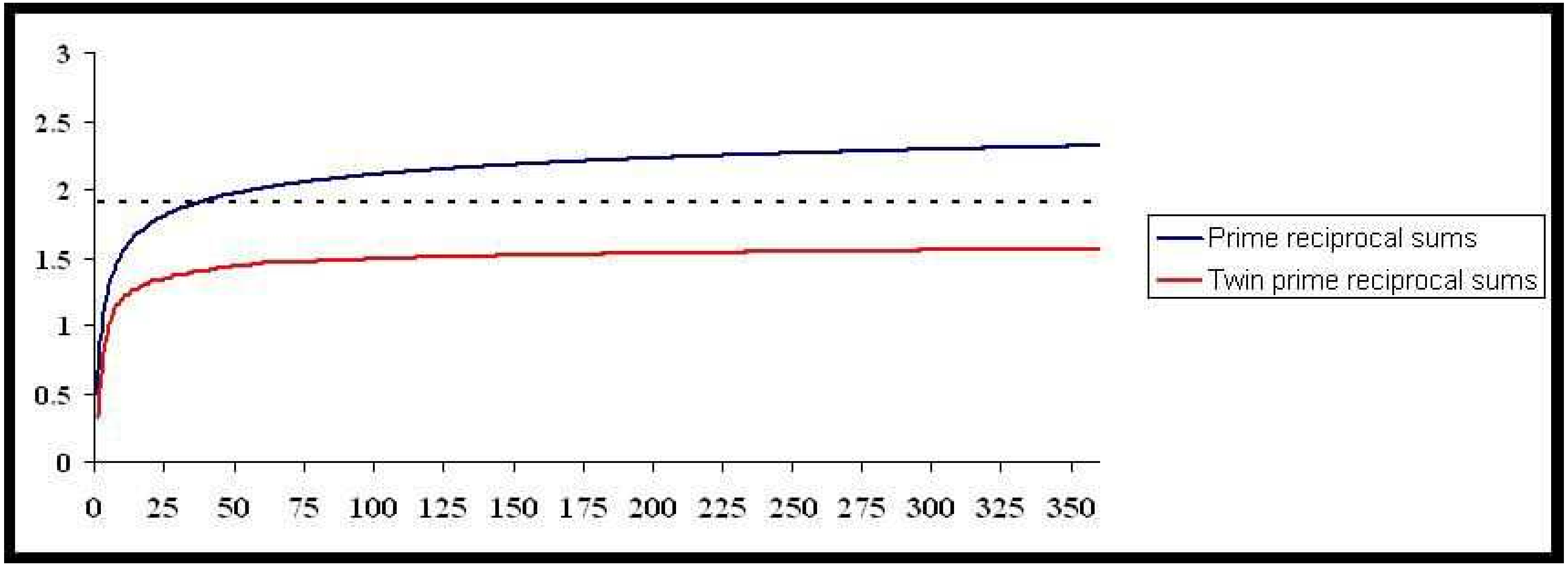
THEOREM OF THE DAY

Brun's Theorem *The sum* $\sum_{\substack{\text{twin } p,q \\ p<q}} \left(\frac{1}{p}+\frac{1}{q}\right)$ *of the reciprocals of twin prime pairs converges.*



Twin primes are those odd prime numbers separated by 2: the smallest possible amount. The first twin prime pair is 3,5; and the summation in the theorem begins $(1/3+1/5)+(1/5+1/7)+(1/11+1/13)+\ldots$. The sum increases as shown in the graph above, in which the horizontal axis covers the first 354 primes, giving all twin prime pairs less than 2500 (the final pair included is 2381, 2383). The lower, red curve increases as these twin prime pairs are progressively included but stays well below the black dotted line to which it will finally converge. This line is Brun's constant and has been estimated at 1.90216 05831 04 by Pascal Sebah using all twin primes less than 10 00000 00000 00000.

The Norwegian Viggo Brun calculated in 1919 that, for $x \geq 3$, the number of twin primes less than x is bounded by $cx(\log\log x)^2/(\log x)^2$, for some constant c, from which the convergence of the sum follows. This result is striking in two ways: (a) if *all* prime reciprocals are added then the sum does *not* converge. Although the upper, black curve in the picture looks identical in shape to the red curve there is no corresponding dotted line of convergence — the black curve will eventually climb above the top of your computer. And (b) Brun proved his theorem even though it is still not known even whether there are infinitely many twin prime pairs. Of course if this, the so-called 'twin primes conjecture', is false then Brun's theorem becomes trivial!

Web link: primes.utm.edu/glossary/page.php/TwinPrime.html

Further reading: *Prime Numbers: The Most Mysterious Figures in Math* by David Wells, John Wiley & Sons, Inc., 1995