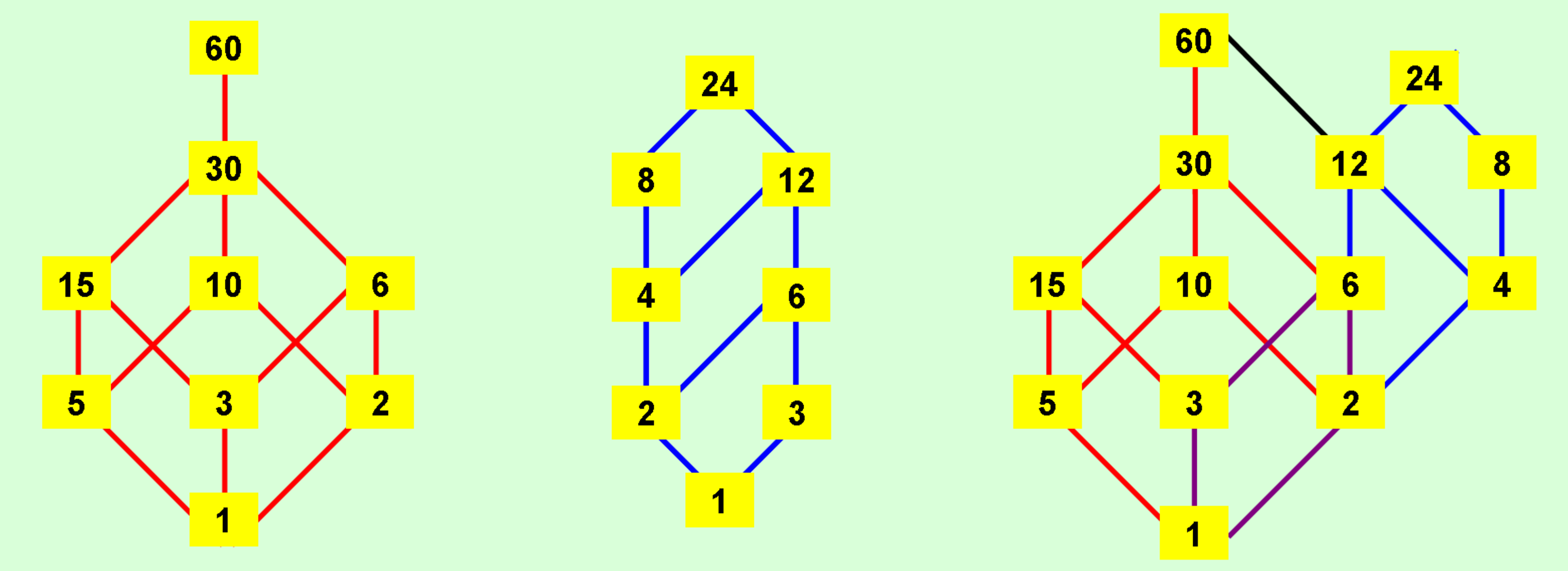
THEOREM OF THE DAY

Dilworth's Theorem *In a finite partial order, the size of a maximum antichain is equal to the minimum number of chains needed to cover its elements.*



An ordering $\leq$ of the elements of a set is a *partial order* if, for any three elements, x, y and z, we have

Reflexivity: $x = y \Rightarrow x \leq y$; **Antisymmetry:** $x \neq y$ and $x \leq y \Rightarrow y \not\leq x$; **Transitivity:** $x \leq y$ and $y \leq z \Rightarrow x \leq z$.

It is perfectly possible for a pair of elements, x and y, to satisfy both $x \not\leq y$ and $y \not\leq x$; such a pair is said to be *incomparable*. In a set partially ordered by the relation "is divisible by", for example, we have both $2 \not\leq 3$ and $3 \not\leq 2$. This is the ordering in the three partially ordered sets (*posets*) shown above, but note that not all pairs that *are* comparable are joined by an edge: from $1 \leq 2$ and $2 \leq 6$, for instance, transitivity automatically implies $1 \leq 6$. These minimally completed diagrams, in which the ordering progresses upwards, are called *Hasse diagrams*.

An *antichain* is any set of mutually incomparable elements; a chain is any set of mutually comparable ones (which includes all ascending paths in a Hasse diagram). Thus, $\{3, 8, 10\}$ is an antichain in the Hasse diagram, above right; $\{8, 10, 12, 15\}$ is a maximum antichain of size 4; and $\{\{5, 15\}, \{1, 10, 30, 60\} \{3, 6, 12\}, \{2, 4, 8, 24\}\}$ is a set of 4 chains which cover all elements.

CALTECH's Robert Dilworth was one of the pioneers of the theory of lattices (posets in which, for any x and y, there are elements w and z, $w \leq x, y$ and $x, y \leq z$). His 1950 theorem may be restated thus: if a poset has $ab + 1$ elements then it has a chain of length $a + 1$ or an antichain of length $b + 1$. In this form it generalises a classic 1935 result of Erdős and Szekeres.

Web link: www.math.gatech.edu/~trotter/Section8-Posets.htm

Further reading: *Introduction to Lattices and Order* by B.A. Davey and H.A. Priestly, Cambridge University Press, 1990.